SERIES REPRESENTATION OF POWER FUNCTION

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Abstract. In this paper we discuss a problem of generalization of binomial distributed triangle, that is sequence A287326 in OEIS. The main property of A287326 that it returns a perfect cube n as sum of n-th row terms over k, such that $0 \leq k \leq n-1$ or $1 \leq k \leq n$, by means of its symmetry. In this paper we have derived a similar triangles in order to receive powers $m = 5, 7$ as row items sum and generalized obtained results in order to receive every odd-powered monomial n^{2m+1}, $m \geq 0$ as sum of row terms of corresponding triangle. In other words, in this manuscript are found and discussed the polynomials $D_m(n,k)$ and $U_m(n,k)$, such that, when being summed up over k in some range with respect to m and n returns the monomial n^{2m+1}.

Contents

1. Structure of the manuscript

The problem of finding expansions of monomials, binomials, trinomials, etc. is classical and a lot of theorems have been found, the most prominent examples are Binomial Theorem [2], Multinomial theorem, Worpitzky Identity [30], [34], Stirling numbers of second kind and falling factorial identity, [36], etc. In this paper we try to solve the classical problem of finding expansions of monomials. We start from binomial distributed triangle A287326, [11] in OEIS. The main property of A287326 that it returns a perfect cube n as n-th row sum, starting from $0, ..., n-1$ or from $1, ..., n$ by means of its symmetry. Therefore, the following question stated:

- Is there a generalization of A287326 in order to receive monomial n^t, $t > 3$ as sum of n-th row terms of corresponding triangle, where t is natural?

Finding an analogs for $t = 5, 7$ in section 3, we answer to above questions positively. Could this process be continued for each odd $t = 1, 3, 5, 7...$ similarly? Positive answer to this question is given by theorem (3.30).

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2. Introduction

Let describe the derivation of the sequence A287326 in OEIS. Sequence A287326 returns the perfect cube n as row sum over $k,\ 0 \leq k \leq n-1$, as well as sum over $1 \leq k \leq n$, by means of its symmetry. First, consider a difference table of perfect cubes ([4], eq. 7)

(2.1)

n	$\Delta^0(n^3)$	$\Delta^1(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	6
6	216	127	42	6
7	343	169	48	6
8	512	217	54	
9	729	271		
10	1000			

Table 1: Difference table of perfect cubes $n,\ 0 \leq n \leq 10$ up to 3^{rd} order.

Reviewing above table, we have noticed that

$$
\begin{aligned}
\Delta(0^3) &= 1+6\cdot 0 = 6\tbinom{1}{2}+\tbinom{1}{0} \\
\Delta(1^3) &= 1+6\cdot 0+6\cdot 1 = 6\tbinom{2}{2}+\tbinom{2}{0} \\
\Delta(2^3) &= 1+6\cdot 0+6\cdot 1+6\cdot 2 = 6\tbinom{3}{2}+\tbinom{3}{0} \\
\Delta(3^3) &= 1+6\cdot 0+6\cdot 1+6\cdot 2+6\cdot 3 = 6\tbinom{4}{2}+\tbinom{4}{0} \\
&\vdots \\
\Delta(n^3) &= 1+6\cdot 0+6\cdot 1+6\cdot 2+\cdots+6\cdot n = 6\tbinom{n+1}{2}+\tbinom{n+1}{0}
\end{aligned} \tag{2.2}
$$

Above difference identity is closely related to Faulhaber's sum of cubes, where $n^3 = 6\binom{n+1}{3}+\binom{n+1}{1}$, see ([21], p. 9). Note that $\Delta^2(n^3)$ could be found similarly using above identity, i.e $\Delta^2(n^3) = 6\binom{n+1}{3-2}+\binom{n+1}{1-2}$.

Property 2.3. *(Generalized finite difference of power using Faulhaber's formula). Consider the identities, (*[21]*, p. 9). For every odd power*

$$
\begin{cases}
n^1 &= \binom{n}{1} \\
n^3 &= 6\binom{n+1}{3}+\binom{n}{1} \\
n^5 &= 120\binom{n+2}{5}+30\binom{n+1}{3}+\binom{n}{1} \\
&\vdots \\
n^{2m-1} &= \sum\limits_{1\leq k\leq m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1}
\end{cases}
$$

The coefficients in these formulas are related to what Riordan [22] *has called central factorial numbers of the second kind. In his notation,*

$$x^m = \sum_{1\leq k\leq m} T(m,k)x^{[k]},\ x^{[k]} = x(x+\tfrac{k}{2}-1)(x+\tfrac{k}{2}-2)\cdots(x+\tfrac{k}{2}+1)$$

The coefficients $T(2m,2k)$ are always integers, because the $x^{[k+2]} = x^{[k]}(x^2/k^4)$ implies the recurrence

$$T(2m+2,2k) = k^2T(2m,2k)+T(2m,2k-2).$$

We can find the first order finite difference of odd power as decreasing the variable of corresponding binomial coefficients by 1, for example

$$\begin{cases} \Delta n^1 & = \binom{n}{0} \\ \Delta n^3 & = 6\binom{n+1}{2} + \binom{n}{0} \\ \Delta n^5 & = 120\binom{n+2}{4} + 30\binom{n+1}{2} + \binom{n}{0} \\ & \vdots \\ \Delta n^{2m-1} & = \sum\limits_{1\le k\le m} (2k-1)!T(2m,2k)\binom{n+k-1}{2k-2} \end{cases}$$

Continue similarly, we can express each difference of order $t \geq 1$. The central factorial numbers of the second kind $(2k-1)!T(2m,2k)$ in above identities are terms of OEIS sequence A303675 and generated by

$$(2k-1)!T(2m,2k) = \frac{1}{r}\sum_{j=0}^{r}(-1)^j\binom{2r}{j}(r-j)^{2n} =: V_{m,k}, \tag{2.4}$$

where $r = n-k+1$, this formula was provided by Peter Luschny in [27]*. Repeated sums are equally easy, in Knuth's notation (see* [21]*, p. 10)*

$$\Sigma^r n^{2m+1} = \sum_{1\le k\le m} V_{m,k}\binom{n+k+r}{2k-1+r}$$

Therefore, reviewing the difference as inverse operator to summation for every odd $t > 0$ and $m \geq 0$, we have identity

$$\Delta^r n^{2m+1} = \sum_{1\le k\le m} V_{m,k}\binom{n+k-r}{2k-1-r}$$

□

By property 2.3 we rewrite the cubes as

$$n^3 = \sum_{1\le k\le n} 6\binom{k+1}{2} + \binom{k}{0} \tag{2.5}$$

Rewrite above expression with set every binomial coefficient to be $\binom{n+1}{2} = 1+2+\cdots+(n+1)$, then

$$n^3 = (1+6\cdot 0) + (1+6\cdot 0+6\cdot 1) + \cdots + (1+6\cdot 0+\cdots+6\cdot(n-1))$$

Particularizing above expression, we get

$$n^3 = n + (n-0)\cdot 6\cdot 0 + (n-1)\cdot 6\cdot 1 + \cdots + (n-(n-1))\cdot 6\cdot(n-1) \tag{2.6}$$

Provided that n is natural. Now, let apply a compact sigma notation on (2.6), thus

$$n^3 = n + \sum_{1\le k\le n} 6k(n-k) \tag{2.7}$$

As sum $\sum_{1\le k\le n} 6k(n-k)$ consists of n terms, we have right to move n in (2.7) under sigma notation, we get

$$n^3 = \sum_{1\le k\le n} 6k(n-k) + 1 \tag{2.8}$$

Property 2.9. *(Proof of symmetry). Let be a sets* $A(n) := \{1,\ 2,\ \ldots,\ n\}$*,* $B(n) := \{0,\ 1,\ \ldots,\ n\}$*,* $C(n) := \{0,\ 1,\ \ldots,\ n-1\}$*, let be expression (2.8) denoted as*

$$M(n,\ C(n)) \ = \ \sum_{k \in C(n)} 6k(n-k)+1$$

where n *is natural-valued variable and* $C(n)$ *is iteration set of (2.8), then we have equality*

$$M(n,\ A(n)) = M(n,\ C(n)) \tag{2.10}$$

Let review and denote expression (2.6) as

$$U(n,\ C(n)) \ = \ n + 6 \cdot \sum_{k \in C(n)} k(n-k)$$

then

$$U(n,\ A(n)) = U(n,\ B(n)) = U(n,\ C(n)) \tag{2.11}$$

Other words, changing of iteration sets of(2.6) and (2.8) by $A(n),\ B(n),\ C(n)$ *and* $A(n),\ C(n)$*, respectively, doesn't change resulting value for each natural* x*.*

Proof. Let be a plot $y(n,k) = 6k(n-k)+1,\ k \in \mathbb{R},\ 0 \le k \le 10$, given $n = 10$

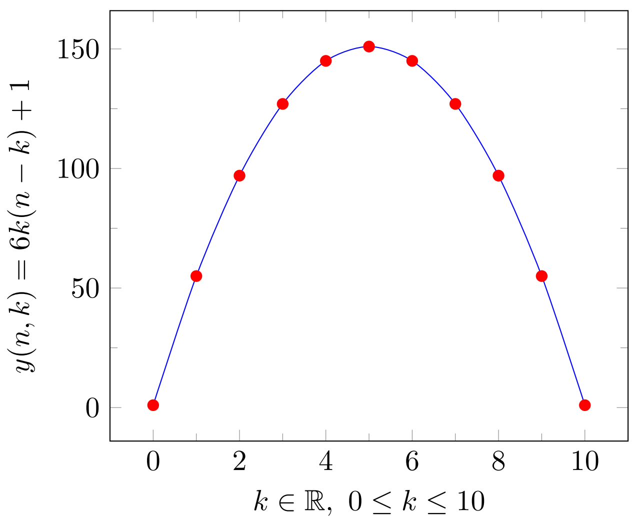


Figure 2. Plot of $6k(n-k)+1,\ k \in \mathbb{R},\ 0 \le k \le n$, where $n = 10$.

Obviously, being a parabolic function, it's symmetrical over $\frac{n}{2}$, hence equivalent $M(n,\ A(n)) = M(n,\ C(n))$ follows. Reviewing (2.6) and denote $u(n,k) = kn - k^2$, we can conclude, that $u(n,0) = u(n,n) = 0$, then equality of $U(n,\ A(n)) = U(n,\ B(n)) = U(n,\ C(n))$ immediately follows. This completes the proof. $\square$

Review above property (2.9). Let be an example of triangle built using

Definition 2.12. For every $n \ge 0$

$$D_1(n,k) \ \overset{\text{def}}{=} \ 6k(n-k)+1,\ 0 \le k \le n \tag{2.13}$$

over n from 0 to $n = 4$, where n denotes corresponding row and k shows the item of row n.

$$
\begin{array}{lccccccccc}
\text{Row 0:} & & & & & 1 & & & & \\
\text{Row 1:} & & & & 1 & & 1 & & & \\
\text{Row 2:} & & & 1 & & 7 & & 1 & & \\
\text{Row 3:} & & 1 & & 13 & & 13 & & 1 & \\
\text{Row 4:} & 1 & & 19 & & 25 & & 19 & & 1
\end{array}
\tag{2.14}
$$

Figure 3. Triangle generated by $D_1(n,k)$, sequence A287326 in OEIS, [11].

Note that n-th row sum of Triangle (2.14) over $0 \leq k \leq n-1$ returns perfect cube n. We can see that each row with respect to variable $n = 0,\ 1,\ 2,\ 3,\ 4, ...,$ has Binomial distribution of row terms. One could compare Triangle (2.14) with Pascal's triangle [1], [12]

$$
\begin{array}{lccccccccc}
\text{Row 0:} & & & & & 1 & & & & \\
\text{Row 1:} & & & & 1 & & 1 & & & \\
\text{Row 2:} & & & 1 & & 2 & & 1 & & \\
\text{Row 3:} & & 1 & & 3 & & 3 & & 1 & \\
\text{Row 4:} & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Figure 4. Pascal's triangle, sequence A007318 in OEIS, [1].

Let us approach to show a few properties of triangle (2.14) and $L_1(n,k)$.

Properties 2.15. *Properties of triangle (2.14).*

(1) *Summation of n-th row of triangle (2.14) over k from 0 to $n-1$ returns perfect cube $n \geq 0$ as follows*

$$n^3 = \sum_{1 \leq k \leq n} D_1(n,k) \tag{2.16}$$

(2) *First item of each row's number corresponding to central polygonal numbers sequence $a(n) = \frac{n^2+n+2}{2}$ (sequence A000124 in OEIS, [13]) returns finite difference of consequent perfect cubes. For example, let be a k-th row of triangle (2.14), such that $k = \frac{n^2+n+2}{2},\ n = 0, 1, 2, ...,$ then item*

$$\Delta(n^3) = D_1\left(\frac{n^2+n+2}{2}, 1\right) \tag{2.17}$$

(3) *Items of (2.14) have Binomial distribution of rows.*

(4) *Linear recurrence, for every k and $n > 0$*

$$2D_1(n,k) = D_1(n+1,k) + D_1(n-1,k) \tag{2.18}$$

This linear recurrence is direct result of second order binomial transform of $D_1(n,k)$ by n.

(5) *Linear recurrence, for each $n > k$*

$$2D_1(n,k) = D_1(2n-k,k) + D_1(2n-k,0) \tag{2.19}$$

(6) *From (1.24) for every $n \geq 0$ follows*

$$n^3 = \sum_{1 \leq k \leq n} D_1(n,k) = \sum_{1 \leq k \leq n} D_1\left(\frac{k^2+k+2}{2}, 1\right) \tag{2.20}$$

(7) *Triangle (2.14) is symmetric, i.e*

$$D_1(n,k) = D_1(n, n-k) \tag{2.21}$$

Corollary 2.22. *Review identity (2.16) in sense of summation of $D_1(n,k)$ over some range W with $\max(W) = T$, then (2.16) returns*

$$\sum_{1\le k\le T} D_1(n,k) = U_1(T,0)n^0 - U_1(T,1)n^1,$$

where $T = 1,2,3,...,\mathbb{N}$. By property (2.9) we rewrite above expression as

$$\sum_{0\le k\le T-1} D_1(n,k) = U_1(T-1,0)n^0 - U_1(T-1,1)n^1$$

Below we show a few initial terms of $U_1(T-1,j),\ U_1(T,j),\ j = 0,1$ coefficients

T	$U_1(T-1,0)$	$U_1(T-1,1)$	$U_1(T,0)$	$U_1(T,1)$
1	1	0	-5	6
2	-4	6	-28	18
3	-27	18	-81	36
4	-80	36	-176	60
5	-175	60	-325	90
6	-324	90	-540	126
7	-539	126	-833	168
8	-832	168	-1216	216
9	-1215	216	-1701	270
10	-1700	270	-2300	330

Table 5. Table of coefficients $U_1(T-1,0), U_1(T-1,1), U_1(T,0), U_1(T,1)$ given $T = 1,...,10$.

Therefore, for every integer $n \ge 1,\ T = n$,

$$\forall T = n:\ n^3 = \begin{cases} U_1(T,0)n^0 - U_1(T,1)n^1, \\ U_1(T-1,0)n^0 - U_1(T-1,1)n^1. \end{cases}$$

Coefficients $|U_1(T-1,1)|$ are terms of sequence A028896 in OEIS, [23]. *Coefficients $|U_1(T,0)|$ are terms of sequence A275709 in OEIS,* [20].

□

In this section we have derived a binomial distributed triangle (2.14), such that perfect cube n could be found as sum of n-th row terms of triangle (2.14) over k, in ranges $1 \le k \le n$ or $0 \le k \le n-1$, where n is natural. Therefore, the follow question is stated:

Question 2.23. *Is there a generalization of A287326 in order to receive monomial $n^t,\ t > 3$ as sum of n-th row terms of corresponding triangle, where t is natural?*

3. Generalization of sequence A287326

In order to get analogs of Triangle (2.14) one should solve a system of equations, where unknowns are coefficients of polynomial and variable of polynomial is $k(n-k)$. Triangle (2.14) is generated by polynomial $D_1(n,k),\ n \ge 0,\ 0 \le k \le n$, defined by (2.12). Here, let derive a triangle generated by polynomial $D_2(n,k),\ n \ge 0,\ 0 \le k \le n$, such that sum of n-th row terms over k, in ranges $1 \le k \le n$ or $0 \le k \le n-1$ returns n^5, where n is natural

Example 3.1. We suspect that n-th row of triangle that returns n^5 as sum of n-th row terms over k, in $1 \le k \le n$ or $0 \le k \le n-1$ is generated by

$$D_2(n,k) = A_{2,2}(n-k)^2k^2 + A_{2,1}(n-k)^1k^1 + A_{2,0}(n-k)^0k^0,\ n \ge 0,\ 0 \le k \le n, \quad (3.2)$$

where $A_{2,2}, A_{2,1}, A_{2,0}$ are unknown coefficients. Assume that for every integer $n \geq 0$ holds

$$n^5 = \sum_{1\leq k\leq n} D_2(n,k). \tag{3.3}$$

To determine the coefficients $A_{2,2}, A_{2,1}, A_{2,0}$, in (3.2) let rewrite (3.3) in extended view

$$\begin{aligned} & A_{2,2}\sum_{1\leq k\leq n} k^2(n-k)^2 + A_{2,1}\sum_{1\leq k\leq n} k(n-k) + \sum_{1\leq k\leq n} A_{2,0} \\ = \; & A_{2,2}\sum_{1\leq k\leq n} k^2(n^2-2nk+k^2) + A_{2,1}\sum_{1\leq k\leq n} kn-k^2 + \sum_{1\leq k\leq n} A_{2,0} \\ = \; & A_{2,2}\sum_{1\leq k\leq n} k^2n^2-2nk^3+k^4 + A_{2,1}\sum_{1\leq k\leq n} kn-k^2 + \sum_{1\leq k\leq n} A_{2,0} \\ = \; & A_{2,2}n^2\sum_{1\leq k\leq n} k^2 - 2A_{2,2}n\sum_{1\leq k\leq n} k^3 + A_{2,2}\sum_{1\leq k\leq n} k^4 + A_{2,1}n\sum_{1\leq k\leq n} k \\ - \; & A_{2,1}\sum_{1\leq k\leq n} k^2 + \sum_{1\leq k\leq n} A_{2,0} = n^5. \end{aligned} \tag{3.4}$$

Thus, we have received expression containing sums of powers of successive natural numbers, where powers are $\{1,2,3,4\}$. These formulas contain so-called Bernoulli numbers, [14]. For mentioned powers of successive natural numbers formulas are following

$$\sum_{1\leq k\leq n} k = \frac{n^2+n}{2}, \tag{3.5}$$

$$\sum_{1\leq k\leq n} k^2 = \frac{2n^3+3n^2+n}{6}, \tag{3.6}$$

$$\sum_{1\leq k\leq n} k^3 = \frac{n^4+2n^3+n^2}{4}, \tag{3.7}$$

$$\sum_{1\leq k\leq n} k^4 = \frac{6n^5+15n^4+10n^3-n}{30}. \tag{3.8}$$

Now we substitute above identities to (3.4), respectively,

$$\begin{aligned} A_{2,2}n^2\frac{2n^3+3n^2+n}{6} \; & - \; 2A_{2,2}n\frac{n^4+2n^3+n^2}{4} + A_{2,2}\frac{6n^5+15n^4+10n^3-n}{30} \\ & + \; A_{2,1}n\frac{n^2+n}{2} - A_{2,1}\frac{2n^3+3n^2+n}{6} + A_{2,0}n \end{aligned}$$

Particularizing the elements of above expression and moving them under the common divisor, we get

$$\frac{A_{2,2}n^5 - A_{2,2}n + 30A_{2,0}}{30} + A_{2,1}\frac{n^3-n}{6} \tag{3.9}$$

We have to remember that expression (3.9) is the left side of the input equation (3.3). Therefore,

$$n^5 = \frac{A_{2,2}n^5 - A_{2,2}n + 30A_{2,0}}{30} + A_{2,1}\frac{n^3-n}{6} \tag{3.10}$$

In order to satisfy (3.10) for each natural n, coefficients $A_{2,0}, A_{2,1}, A_{2,2}$ should be a solutions of following system of equations

$$\begin{cases} \frac{1}{30}A_{2,2} & = 1 \\ A_{2,1} & = 1 \\ 30A_{2,0} - A_{2,2} & = 0 \end{cases}$$

The only solution of above system is $A_{2,2} = 30,\ A_{2,1} = 0,\ A_{2,0} = 1$. Hereby, polynomial $D_2(n,k)$ takes the form

$$D_2(n,k) = A_{2,2}(n-k)^2k^2 + A_{2,1}(n-k)^1k^1 + A_{2,0}(n-k)^0k^0 = 30k^2(n-k)^2 + 1 \tag{3.11}$$

And for every natural $n \geq 1$ holds

$$\begin{aligned} n^5 &= \sum_{1 \leq k \leq n} D_2(n,k) = \sum_{1 \leq k \leq n} 30k^2(n-k)^2 + 1 \\ &= \sum_{0 \leq k \leq n-1} D_2(n,k) = \sum_{0 \leq k \leq n-1} 30k^2(n-k)^2 + 1 \end{aligned} \tag{3.12}$$

Let show few initial rows of triangle built by $D_2(n,k)$, that is analog of triangle (2.14), which returns monomial n^5 as sum of n-th row terms over k, as $1 \leq k \leq n$ or $0 \leq k \leq n-1$, by means of its symmetry

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 31 & & 1 & & & \\ & & 1 & & 121 & & 121 & & 1 & & \\ & 1 & & 271 & & 481 & & 271 & & 1 & \\ 1 & & 480 & & 1081 & & 1081 & & 481 & & 1 \end{array} \tag{3.13}$$

Figure 6. Triangle generated by polynomial $D_2(n,k),\ n \geq 0,\ 0 \leq k \leq n$, sequence A300656 in OEIS, [15].

Similarly, finding the coefficients $A_{3,0}, A_{3,1}, A_{3,2}, A_{3,3}$ in

$$D_3(n,k) = A_{3,3}k^3(n-k)^3 + A_{3,2}k^2(n-k)^2 + A_{3,1}k^1(n-k)^1 + A_{3,0}k^0(n-k)^0, \tag{3.14}$$

we get $A_{3,3} = 140,\ A_{3,2} = -14,\ A_{3,1} = 0,\ A_{3,0} = 1$, therefore, for each integer $n \geq 0$ holds

$$\begin{aligned} n^7 &= \sum_{1 \leq k \leq n} D_3(n,k) = \sum_{1 \leq k \leq n} 140k^3(n-k)^3 - 14k^2(n-k)^2 + 1 \\ &= \sum_{0 \leq k \leq n-1} D_3(n,k) = \sum_{0 \leq k \leq n-1} 140k^3(n-k)^3 - 14k^2(n-k)^2 + 1 \end{aligned} \tag{3.15}$$

Below we show a few initial rows of triangle generated by polynomial $D_3(n,k),\ n \geq 0,\ 0 \leq k \leq n$, the analog of triangle (2.14), such that monomial n^7 could be found as sum of n-th row terms over k, as $1 \leq k \leq n$ or $0 \leq k \leq n-1$, by means of its symmetry

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 127 & & 1 & & & \\
 & & 1 & & 1093 & & 1093 & & 1 & & \\
 & 1 & & 3793 & & 8905 & & 3793 & & 1 & \\
1 & & 8905 & & 30157 & & 30157 & & 8905 & & 1
\end{array} \tag{3.16}
$$

Figure 7. Triangle generated by polynomial $D_3(n,k),\ n \geq 0,\ 0 \leq k \leq n$, sequence A300785 in OEIS, [16].

We assume now that generalization of A287326 holds for odd powers of the form $2m+1,\ m = 0,1,2,...$, where A287326 is partial case for $m=1$. To generalize our sequences A287326, A300656, A300785 for every odd power $2m+1,\ m \geq 0$ one has to review the generating functions of sequences A287326, A300656, A300785 as follows. Let be definition

Definition 3.17.

$$
\begin{aligned}
D_m(n,k) \ &:= \ A_{m,m}k^m(n-k)^m + A_{m,m-1}k^{m-1}(n-k)^{m-1} + \cdots + A_{m,0}k^0(n-k)^0 \\
&= \sum_{0 \leq j \leq m} A_{m,j}k^j(n-k)^j,
\end{aligned}
$$

where $A_{m,j},\ 0 \leq j \leq m$ are unknown coefficients.

And, we assume that

$$
n^{2m+1} = \sum_{1 \leq k \leq n} D_m(n,k). \tag{3.18}
$$

We want to notice that as we used a compact sigma notation on definition (3.17), i.e we rewrite (3.17) as $\sum_{0 \leq j \leq m} A_{m,j}k^j(n-k)^j$, thus the sum (3.18) returns indeterminate form of $D_m(n,k) = \sum_{0 \leq j \leq m} A_{m,j}k^j(n-k)^j$ on step $k=n$ as $A_{m,0}(n-n)^0k^0$ contains the term $(n-n)^0 = 0^0$. Some textbooks leave the quantity 0^0 undefined, because the functions x^0 and 0^x have different limiting values when x decreases to 0. But this is a mistake. We must define

$$
\forall x: \ x^0 = 1,
$$

if the binomial theorem is to be valid when $x=0,\ y=0$, and/or $x=y$. The binomial theorem is too important to be arbitrarily restricted! By contrast, the function 0^x is quite unimportant, [31]. Note that $D_m(n,k)$ is generalization of (2.12) and (3.11). For example, generating functions of sequences A287326, A300656, A300785 are, respectively

$$
\begin{cases}
D_1(n,k) = 1 + 6k(n-k), & \text{for } A287326 \\
D_2(n,k) = 1 - 0k(n-k) + 30k^2(n-k)^2, & \text{for } A300656 \\
D_3(n,k) = 1 - 14k(n-k) + 0k^2(n-k)^2 + 140k^3(n-k)^3, & \text{for } A300785
\end{cases}
$$

Where coefficients $A_{m,j}$, for $m = 1,2,3$ are $\{A_{1,j}\}_{j=0}^{1} = \{1,6\},\ \{A_{2,j}\}_{j=0}^{2} = \{1,0,30\},\ \{A_{3,j}\}_{j=0}^{3} = \{1,-14,0,140\}$ in definitions of generating functions of A287326, A300656, A300785. To generalize above result in order to receive monomial n^{2m+1} as $\sum_{1 \leq k \leq n} D_m(n,k) = n^{2m+1},\ m = 0,1,2,...$ one has to solve the system of equations. Complete set of coefficients $\{A_{m,0},\ \ldots,\ A_{m,m}\}$ such that

$\sum_{1\leq k\leq n} D_m(n,k) = n^{2m+1},\ m\geq 0$ holds can be found by solving the following system of equations

$$
\begin{cases}
D_m(1,0) = 1^{2m+1}\\
D_m(2,0)+D_m(2,1) = 2^{2m+1}\\
D_m(3,0)+D_m(3,1)+D_m(3,2) = 3^{2m+1}\\
\vdots\\
D_m(r,0)+D_m(r,1)+\cdots+D_m(r,r-1) = r^{2m+1},\ r>m
\end{cases}
\tag{3.19}
$$

List of solutions[1] of system (3.19) is split and assigned to OEIS under the numbers A302971 (numerators of $A_{m,j}$) and A304042 (denominators of $A_{m,j}$). To reach recurrent formula of $A_{m,j}$, first let fix the unused values $A_{m,j}=0$, for $j<0$ or $j>m$, so we don't need to care about the summation range for j, then by expanding $(n-k)^j$ and using Faulhaber's formula [7], we get

$$
\begin{aligned}
\sum_{k=0}^{n-1}(n-k)^j k^j &= \sum_{k=0}^{n-1}\sum_{i}^{\infty}\binom{j}{i}n^{j-i}(-1)^i k^{i+j} \qquad (3.20)\\
&= \sum_{i}^{\infty}\binom{j}{i}n^{j-i}\frac{(-1)^i}{i+j+1}\left[\sum_{t}^{\infty}\binom{i+j+1}{t}B_t n^{i+j+1-t} - B_{i+j+1}\right]\\
&= \underbrace{\sum_{i,t}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t}B_t n^{2j+1-t}}_{(\star)} - \underbrace{\sum_{i}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}B_{i+j+1}n^{j-i}}_{(\diamond)}
\end{aligned}
$$

where B_t are Bernoulli numbers [14]. Now, we notice that

$$
\sum_{i}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t} = \begin{cases}\frac{1}{(2j+1)\binom{2j}{j}}, & \text{if } t=0;\\ \frac{(-1)^j}{t}\binom{j}{2j-t+1}, & \text{if } t>0\end{cases}
\tag{3.21}
$$

In particular, the last sum is zero for $0<t\leq j$. Now we revise the $(\star)$ part of (3.20) according to results of (3.21), thus

$$
\begin{aligned}
\sum_{i,t}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t}B_t n^{2j+1-t} &= \frac{1}{(2j+1)\binom{2j}{j}}\\
&+ \sum_{t>0}\frac{(-1)^j}{t}\binom{j}{2j-t+1}B_t n^{2j+1-t}
\end{aligned}
$$

Therefore, (3.20) takes the form

$$
\begin{aligned}
\sum_{k=0}^{n-1}(n-k)^j k^j &= \underbrace{\frac{1}{(2j+1)\binom{2j}{j}} + \sum_{t>0}\frac{(-1)^j}{t}\binom{j}{2j-t+1}B_t n^{2j+1-t}}_{(\star)} \qquad (3.22)\\
&- \underbrace{\sum_{i}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}B_{i+j+1}n^{j-i}}_{(\diamond)}
\end{aligned}
$$

Now, we keep our attention to (3.22) and we have to remember that if the sum over some variable i contains $\binom{j}{i}$, then instead of limiting its summation range to $i=0,...,j$, we can let $i=-\infty,...,+\infty$

[1] solutions_system_2_4.txt - Mathematica code, provides the list of solutions of system (3.19) up to $r=11$, [24].

since $\binom{j}{i} = 0$ for i outside the range $i = 0, ..., j$ (i.e., when $i < 0$ or $i > j$). It's much easier to review such sum as summing from $-\infty$ to $+\infty$ (unless specified otherwise), where only a finite number of terms are nonzero, this fact is discussed in [28] as well. To combine or cancel identical terms across the two sums in (3.22) more easily, we introduce $\ell = 2j + 1 - t$ to $(\star)$ and $\ell = j - i$ to $(\diamond)$, respectively, we get

$$\begin{aligned}\sum_{k=0}^{n-1}(n-k)^j k^j &= \frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1} + \sum_{\ell=-\infty}^{\infty}\frac{(-1)^j}{2j+1-\ell}\binom{j}{\ell}B_{2j+1-\ell}n^{\ell} \\ &- \sum_{\ell=-\infty}^{\infty}\binom{j}{\ell}\frac{(-1)^{j-\ell}}{2j+1-\ell}B_{2j+1-\ell}n^{\ell} \\ &= \frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1} + 2\sum_{\text{odd }\ell}^{\infty}\frac{(-1)^j}{2j+1-\ell}\binom{j}{\ell}B_{2j+1-\ell}n^{\ell}.\end{aligned} \tag{3.23}$$

Now, using the definition of $A_{m,j}$, we obtain the following identity for polynomials in n

$$\begin{aligned}&\sum_{j}^{\infty} A_{m,j}\frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1} + 2\sum_{j,\text{ odd }\ell}^{\infty} A_{m,j}\binom{j}{\ell}\frac{(-1)^j}{2j+1-\ell}B_{2j+1-\ell}n^{\ell} \\ &\equiv n^{2m+1}.\end{aligned} \tag{3.24}$$

Taking the coefficient of n^{2m+1} in above expression, we get $A_{m,m} = (2m+1)\binom{2m}{m}$, and taking the coefficient of x^{2d+1} for an integer d in the range $m/2 \le d < m$ we get $A_{m,d} = 0$. Taking the coefficient of n^{2d+1} in (3.24) for $m/4 \le d < m/2$, we get

$$A_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0, \tag{3.25}$$

i.e

$$A_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}. \tag{3.26}$$

Continue similarly, we can express $A_{m,j}$ for each integer j in range $m/2^{s+1} \le j < m/2^s$ (iterating consecutively $s = 1, 2, ...$) via previously determined values of $A_{m,d},\ d < j$ as follows

$$A_{m,j} = (2j+1)\binom{2j}{j}\sum_{d=2j+1}^{m} A_{m,d}\binom{d}{2j+1}\frac{(-1)^{d-1}}{d-j}B_{2d-2j}. \tag{3.27}$$

The same formula holds also for $m = 0$. Note that in above sum m have to be $m \ge 2j+1$ to return nonzero term $A_{m,j}$.

Definition 3.28. We define here a generating function of sequence of coefficients $A_{m,j}$, such that $\sum_{k=0}^{n-1} D_m(n,k) = n^{2m+1},\ n \ge 0,\ m \ge 0$, where $D_m(n,k)$ is defined by (3.17)

$$A_{m,j} := \begin{cases} 0, & \text{if } j < 0 \text{ or } j > m \\ (2j+1)\binom{2j}{j}\sum_{d=2j+1}^{m} A_{m,d}\binom{d}{2j+1}\frac{(-1)^{d-1}}{d-j}B_{2d-2j}, & \text{if } 0 \le j < m \\ (2j+1)\binom{2j}{j}, & \text{if } j = m \end{cases}$$

Five initial rows of triangle generated by $A_{m,j},\ j \ge 0,\ 0 \le j \le m$ are

$$
\begin{array}{lccccccccccc}
m=0 & & & & & & 1 & & & & & \\
m=1 & & & & & 1 & & 6 & & & & \\
m=2 & & & & 1 & & 0 & & 30 & & & \\
m=3 & & & 1 & & -14 & & 0 & & 140 & & \\
m=4 & & 1 & & -120 & & 0 & & 0 & & 630 & \\
m=5 & 1 & & -1386 & & 660 & & 0 & & 0 & & 2772
\end{array} \tag{3.29}
$$

Figure 8. Triangle generated by $A_{m,j},\ j \geq 0,\ 0 \leq j \leq m$, sequences A302971 (numerators of $A_{m,j}$) and A304042 (denominators of $A_{m,j}$).

Note that starting from row $m \geq 11$ the terms of Triangle (3.29) consist fractional numbers, for example, $A_{11,1} = 800361655623, 6$. One can find complete list of the numerators and denominators of $A_{m,j}$ in OEIS under the identifiers A302971 and A304042, respectively, see [17],[18]. To verify the terms that definition (3.28) produces one should refer to Mathematica code[2]. Hereby, let be theorem

Theorem 3.30. *For every non-negative integers $n \geq 0$ and $m \geq 0$ holds*

$$
n^V = \begin{cases} \sum\limits_{1 \leq k \leq n} D_m(n,k) = \sum\limits_{1 \leq k \leq n} \sum\limits_{0 \leq j \leq m} A_{m,j} k^j (n-k)^j, & \text{for } V = 2m+1 \\ \sum\limits_{1 \leq k \leq n} \frac{1}{n} D_m(n,k) = \sum\limits_{1 \leq k \leq n} \frac{1}{n} \sum\limits_{0 \leq j \leq m} A_{m,j} k^j (n-k)^j, & \text{for } V = 2m, \end{cases}
$$

where $D_m(n,k)$ is defined by (3.17).

One can verify results concerning above theorem (3.30) via Mathematica code[3]. Therefore, theorem (3.30) answers to the question (2.23) positively, since for every $m \geq 0$ exists a triangle, generated by $D_m(n,k),\ n \geq 0,\ 0 \leq k \leq n$, such that odd power n^{2m+1} can be reached as sum of n-th row of corresponding triangle over $1 \leq k \leq n$ or $0 \leq k \leq n-1$. Sequences A287326, A300656, A300785 are partial cases of theorem (3.30) for $m = 1, 2, 3$, respectively.

3.1. **Properties of $D_m(n,k)$ and $A_{m,j}$.** Here we show a few properties of definition $D_m(n,k)$, some of them correlates with properties of partial case $D_1(n,k)$ in 2.15.

(1) Sum of $A_{m,j}$ over j in range $0 \leq j \leq m$ gives
$$\sum_{0 \leq j \leq m} A_{m,j} \ = \ 2^{2m+1} - 1,$$
where $A_{m,j}$ is defined by (3.28).

(2) Similarly to property (2.21) of particular case $D_1(n,k)$, items of $\{D_m(n,k)\}_{k=0}^{n},\ m \geq 0,\ n \geq 0$ is symmetric, i.e
$$D_m(n,k) = D_m(n, n-k),$$
for all $k:\ 0 \leq k \leq n$.

(3) By property (2) for every integer $n \geq 0,\ m \geq 0$ immediately follows
$$n^{2m+1} = \sum_{1 \leq k \leq n} D_m(n,k) \ = \ \sum_{0 \leq k \leq n-1} D_m(n,k)$$

(4) For every $m \geq 0$ the $A_{m,m}$ are terms of the sequence A002457, [19].

(5) For each $m \geq 0$
$$A_{m,0} \ = \ 1$$

[2]def_2_12.txt, - Mathematica code, implementation of definition (3.28), [25].

[3]expression_2_1.txt, - Mathematica code, implementation of theorem (3.30), [26].

(6) Assume that $n < 0$, then theorem (3.30) can be applied as

$$n^V = \begin{cases} \sum\limits_{1\leq k\leq |n|} D_m(n,k), & \text{for } V = 2m+1 \\ \sum\limits_{1\leq k\leq |n|} \frac{1}{|n|} D_m(|n|,k), & \text{for } V = 2m \end{cases}$$

Property 3.31. *(Linear Recurrence of $D_m(n,k)$.) For every integer $n \geq 0$ in $D_1(n,k)$, $D_2(n,k)$, $D_3(n,k)$ hold the recurrent relations*

$$\begin{aligned} D_1(n+1,k) &= 2D_1(n,k) - D_1(n-1,k) \\ D_2(n+2,k) &= 3D_2(n+1,k) - 3D_2(n,k) + D_2(n-1,k) \\ D_3(n+3,k) &= 4D_3(n+2,k) - 6D_3(n+1,k) + 4D_3(n,k) - D_3(n-1,k) \end{aligned}$$

Review the coefficient $D_m(n,k)$, which defined by m-order polynomial. It's well known fact that high order finite difference $\Delta^{m+k} P_m(x) = 0$ for every x, where $P_m(x)$ is m order polynomial and $k > 0$, [8]. *Recall Binomial Transform of sequence a_n. D. E. Knuth* [35] *has introduced the binomial transform by*

$$\hat{a}_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k, \tag{3.32}$$

In particular, $\hat{a}_n = \Delta^n a_n$, therefore, for $D_m(n,k)$ we have

$$\forall t \geq m: \ \widehat{D}_m(n,k)_t = \sum_{j\geq 0} \left[(-1)^j \binom{t+1}{j} D_m(n+t-j,k) \right] \equiv 0, \tag{3.33}$$

hereby, it gives us right to represent recursively every value of $D_m(n+r,k)$, $0 \leq r \leq t+1$ as

$$(-1)^{r+1} \binom{t+1}{r} D_m(n+t-r,k) = \sum_{j\in\mathbb{Z}_{\geq 0}/\{r\}} (-1)^{j+1} \binom{t+1}{j} D_m(n+t-j,k) \tag{3.34}$$

In particular,

$$D_m(n+t,k) = \sum_{j\geq 1} (-1)^{j+1} \binom{t+1}{j} D_m(n+t-j,k) \tag{3.35}$$

□

Hereby, let be theorem

Theorem 3.36. *By property (3.31), particularly, from expression (3.35), for every integer $t \geq m$ follows*

$$(n+t)^{2m+1} = \sum_{k\geq 1} \sum_{j\geq 1} (-1)^{j+1} \binom{t+1}{j} D_m(n+t-j,k) \tag{3.37}$$

Proof. Direct consequence of theorem (3.30) and (3.35). □

3.2. **Example of use of theorem (3.30).** In this subsection we show a detailed application of theorem (3.30). In this subsection we highlight the corresponding terms of $A_{m,j}$, $0 \leq j \leq m$ and $T(n,k)$, $1 \leq k \leq n$ with different colors to be more easily to see regularity. Recall existing pattern, that is triangle of coefficients $A_{m,j}$ defined by (3.28)

$$\begin{array}{lccccccccccc}
m=0 & & & & & & 1 & & & & & \\
m=1 & & & & & 1 & & 6 & & & & \\
m=2 & & & & 1 & & 0 & & 30 & & & \\
m=3 & & & 1 & & -14 & & 0 & & 140 & & \\
m=4 & & 1 & & -120 & & 0 & & 0 & & 630 & \\
m=5 & 1 & & -1386 & & 660 & & 0 & & 0 & & 2772
\end{array}$$

Figure 9. Triangle generated by $A_{m,j},\ m \geq 0,\ 0 \leq j \leq m$.

By received formula $\sum_{1\leq k\leq n}\sum_{j\geq 0} D_m(n,k) = n^{2m+1}$ each line of above triangle being multiplied by $T^{j>0}(n,k) := k^j(n-k)^j$ and summed up to n or $n-1$ over k from 0 or 1, respectively, will result odd power of n^{2m+1}, depending on the row $A_{m,j},\ 0 \leq j \leq m$ is applied. Consider the case $n=3,\ m=2$, we introduce triangle built using $T(n,k) := k(n-k),\ n \geq 1,\ 1 \leq k \leq n$ as upper triangular array, let be $1 \leq n \leq 5,\ 1 \leq k \leq n$

$$\begin{array}{ccccc}
n=1 & n=2 & n=3 & n=4 & n=5 \\
\hline
0 & 1 & 2 & 3 & 4 \\
 & 0 & \underline{2} & 4 & 6 \\
 & & 0 & 3 & 6 \\
 & & & 0 & 4 \\
 & & & & 0
\end{array} \tag{3.38}$$

Figure 10. Upper right triangle generated by $T(n,k) = k(n-k),\ n \geq 1,\ 1 \leq k \leq n$, sequence A094053, [29] in OEIS.

Example 3.39. Consider theorem (3.30), let be $n=3$ and $m=2$, then

$$\begin{aligned}
3^{2\cdot 2+1} &= 1\cdot 2^0 + 0\cdot 2^1 + 30\cdot 2^2 \\
&+ 1\cdot \underline{2}^0 + 0\cdot \underline{2}^1 + 30\cdot \underline{2}^2 \\
&+ 1\cdot 0^0 + 0\cdot 0^1 + 30\cdot 0^2 \\
&= 121 + 121 + 1 = 243
\end{aligned}$$

Items in above array are terms of the third row of triangle A300656.

Example 3.40. Consider theorem (3.30), let be $n=3$ and $m=3$, then

$$\begin{aligned}
3^{2\cdot 3+1} &= 1\cdot 2^0 - 14\cdot 2^1 + 0\cdot 2^2 + 140\cdot 2^3 \\
&+ 1\cdot 2^0 - 14\cdot 2^1 + 0\cdot 2^2 + 140\cdot 2^3 \\
&+ 1\cdot 0^0 - 14\cdot 0^1 + 0\cdot 0^2 + 140\cdot 0^3 \\
&= 1093 + 1093 + 1 = 2187
\end{aligned}$$

Items in above array are terms of the third row of triangle A300785.

Example 3.41. Consider theorem (3.30), let be $n=4$ and $m=3$, then

$$\begin{aligned}
4^{2\cdot 3+1} &= 1\cdot 3^0 - 14\cdot 3^1 + 0\cdot 3^2 + 140\cdot 3^3 \\
&+ 1\cdot 4^0 - 14\cdot 4^1 + 0\cdot 4^2 + 140\cdot 4^3 \\
&+ 1\cdot 3^0 - 14\cdot 3^1 + 0\cdot 3^2 + 140\cdot 3^3 \\
&+ 1\cdot 0^0 - 14\cdot 0^1 + 0\cdot 0^2 + 140\cdot 0^3 \\
&= 3739 + 8905 + 3739 + 1 = 16384
\end{aligned}$$

Items in above array are terms of the forth row of triangle A300785. We can perform same result for $4^{2\cdot3+1}$ in terms of multiplication of certain matrices,

$$4^{2\cdot3+1}=[1,1,1,1]\cdot\begin{bmatrix}3^0 & 3^1 & 3^2 & 3^3\\ 4^0 & 4^1 & 4^2 & 4^3\\ 3^0 & 3^1 & 3^2 & 3^3\\ 0^0 & 0^1 & 0^2 & 0^3\end{bmatrix}\cdot\begin{bmatrix}1\\ -14\\ 0\\ 140\end{bmatrix}$$

Example 3.42. Consider theorem (3.30), let be $n=5$ and $m=3$, then

$$\begin{aligned}5^{2\cdot3+1} &= 1\cdot4^0-14\cdot4^1+0\cdot4^2+140\cdot4^3\\ &+ 1\cdot6^0-14\cdot6^1+0\cdot6^2+140\cdot6^3\\ &+ 1\cdot6^0-14\cdot6^1+0\cdot6^2+140\cdot6^3\\ &+ 1\cdot4^0-14\cdot4^1+0\cdot4^2+140\cdot4^3\\ &+ 1\cdot0^0-14\cdot0^1+0\cdot0^2+140\cdot0^3\\ &= 8905+30157+30157+8905+1=78125\end{aligned}$$

Items in above array are terms of the fifth row of triangle A300785.

Example 3.43. Consider theorem (3.30), let be $n=5$ and $m=4$, then

$$\begin{aligned}5^{2\cdot4+1} &= 1\cdot4^0-120\cdot4^1+0\cdot4^2+0\cdot4^3+630\cdot4^4\\ &+ 1\cdot6^0-120\cdot6^1+0\cdot6^2+0\cdot6^3+630\cdot6^4\\ &+ 1\cdot6^0-120\cdot6^1+0\cdot6^2+0\cdot6^3+630\cdot6^4\\ &+ 1\cdot4^0-120\cdot4^1+0\cdot4^2+0\cdot4^3+630\cdot4^4\\ &+ 1\cdot0^0-120\cdot0^1+0\cdot0^2+0\cdot0^3+630\cdot0^4\\ &= 160801+815761+815761+160801+1=1953125\end{aligned}$$

4. Another power identity

Review the Corollary (2.2), it says that partial case of theorem (3.30) returns the binomial of the form

$$\begin{aligned}\sum_{M\le k\le N} D_1(n,k) &= \begin{cases}U_1(T,0)n^0+U_1(T,1)n^1, & \text{if } M=1,\ N=T\\ U_1(T-1,0)n^0+U_1(T-1,1)n^1, & \text{if } M=0,\ N=T-1\end{cases}\\ &= n^3,\ \text{as } T\to n.\end{aligned}$$

Let extend this idea, as we have complete theorem (3.30). We rewrite theorem (3.30) for odd powers as

$$\begin{aligned}\sum_{M\le k\le N} D_m(n,k) &= \begin{cases}U_m(T,0)n^0+\cdots+U_m(T,m)n^m, & \text{if } M=1,\ N=T\\ U_m(T-1,0)n^0+\cdots+U_m(T-1,m)n^m, & \text{if } M=0,\ N=T-1\end{cases}\\ &= n^{2m+1},\ \text{as } T=n.\end{aligned}$$

Above expression is so-called 'closed form' of the theorem (3.30) for every particular n. Below we place a few examples of polynomials $\sum_{0\leq k\leq m}(-1)^{m-k}U_m(T,k)\cdot n^k$, where $m=2$,

(4.1)

T	$U_2(T,0)n^0+U_2(T,1)n^1+U_2(T,2)n^2$
1	$31-60n+30n^2$
2	$512-540n+150n^2$
3	$2943-2160n+420n^2$
4	$10624-6000n+900n^2$
5	$29375-13500n+1650n^2$
6	$68256-26460n+2730n^2$
7	$140287-47040n+4200n^2$
8	$263168-77760n+6120n^2$
9	$459999-121500n+8550n^2$
10	$760000-181500n+11550n^2$

Figure 11. Table of polynomials generated by $\sum_{1\leq k\leq T}D_2(n,k)=n^5$ as $T\to n$, see definition (3.17) for $D_2(n,k)$.

Below we show a plots of few first polynomials $\sum_{1\leq k\leq T}D_2(n,k)$ for $T=1,2,3$ and compare it with corresponding monomial n^5 by the theorem (3.30),

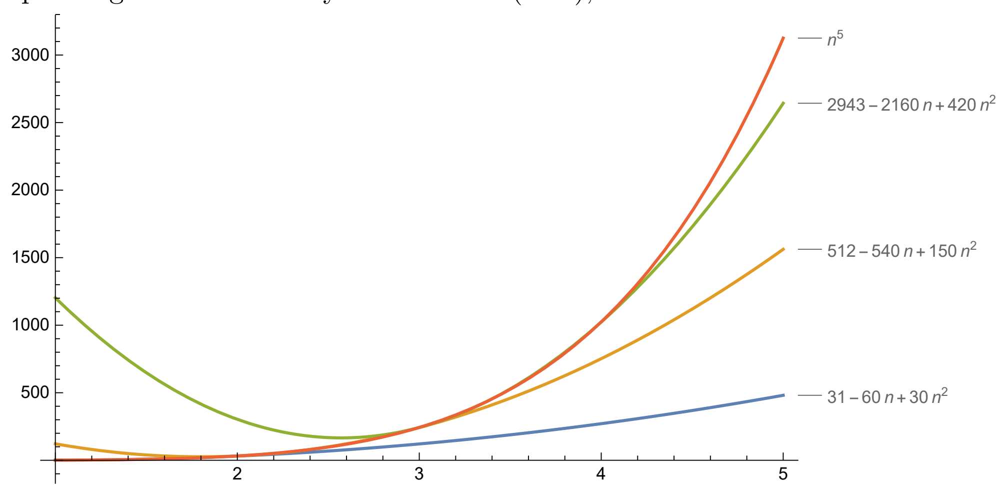


Figure 12. Local approximations of monomial n^5 by corresponding polynomials $\sum_{1\leq k\leq T}D_2(n,k)$ for $T=1,2,3$, by theorem (3.30).

We can see that monomial n^5 can be easily approximated in neighborhood of every particular natural n. The polynomials from figure (11) can be generated using Mathematica code[4]. To understand the nature of coefficients $U_m(T,k)$, let derive them directly by means of theorem (3.30),

$$\begin{aligned}
(4.2)\qquad \sum_{k=1}^{n}\sum_{j=0}^{m}A_{m,j}k^j(n-k)^j &= \sum_{k=1}^{n}A_{m,0}k^0(n-k)^0+\cdots+A_{m,m}k^m(n-k)^m \\
&= \sum_{k=1}^{n}A_{m,0}k^0(n-k)^0+\sum_{k=1}^{n}A_{m,1}k^1(n-k)^1+\cdots+\sum_{k=1}^{n}A_{m,m}k^m(n-k)^m \\
&= A_{m,0}\sum_{k=1}^{n}k^0(n-k)^0+A_{m,1}\sum_{k=1}^{n}k^1(n-k)^1+\cdots+A_{m,m}\sum_{k=1}^{n}k^m(n-k)^m \\
&= n^{2m+1}.
\end{aligned}$$

[4]Um(n,k)_coefficients2.txt. - Mathematica code, generates the polynomials from figure (11), [39]. If necessary, one could re-define $m\geq 0$ within the code.

By the binomial theorem,

$$\sum_{k=0}^{n-1}(n-k)^j k^j = \sum_{k=0}^{n-1}\sum_{i=0}\binom{j}{i} n^{j-i}(-1)^i k^{j+i}$$

We rewrite the main result of (4.2) as

$$\begin{aligned}
(4.3)\qquad & A_{m,0}\sum_{k=1}^{n} k^0(n-k)^0 + A_{m,1}\sum_{k=1}^{n} k^1(n-k)^1 + \cdots + A_{m,m}\sum_{k=1}^{n} k^m(n-k)^m \\
= \quad & A_{m,0}\sum_{k=1}^{n}\sum_{i=0}\binom{0}{i} n^{0-i}(-1)^i k^{0+i} + A_{m,1}\sum_{k=1}^{n}\sum_{i=0}\binom{1}{i} n^{1-i}(-1)^i k^{1+i} \\
+ \quad & A_{m,2}\sum_{k=1}^{n}\sum_{i=0}\binom{2}{i} n^{2-i}(-1)^i k^{2+i} + A_{m,3}\sum_{k=1}^{n}\sum_{i=0}\binom{3}{i} n^{3-i}(-1)^i k^{3+i} \\
\vdots \quad & \\
+ \quad & A_{m,m}\sum_{k=1}^{n}\sum_{i=0}\binom{m}{i} n^{m-i}(-1)^i k^{m+i} = n^{2m+1}
\end{aligned}$$

Rewrite expression (4.3) in extended view

$$\begin{aligned}
(4.4)\qquad & A_{m,0}\sum_{k=1}^{n} k^0(n-k)^0 + A_{m,1}\sum_{k=1}^{n} k^1(n-k)^1 + \cdots + A_{m,m}\sum_{k=1}^{n} k^m(n-k)^m \\
= \quad & A_{m,0}\sum_{k=1}^{n}\left[\binom{0}{0} n^0(-1)^0 k^0\right] + A_{m,1}\sum_{k=1}^{n}\left[\binom{1}{0} n^1(-1)^0 k^1 + \binom{1}{1} n^0(-1)^1 k^2\right] \\
+ \quad & A_{m,2}\sum_{k=1}^{n}\left[\binom{2}{0} n^2(-1)^0 k^2 + \binom{2}{1} n^1(-1)^1 k^3 + \binom{2}{2} n^0(-1)^2 k^4\right] \\
+ \quad & A_{m,3}\sum_{k=1}^{n}\left[\binom{3}{0} n^3(-1)^0 k^3 + \binom{3}{1} n^2(-1)^1 k^4 + \binom{3}{2} n^1(-1)^2 k^5 + \binom{3}{3} n^0(-1)^3 k^6\right] \\
\vdots \quad & \\
+ \quad & A_{m,m}\sum_{k=1}^{n}\left[\binom{m}{0} n^m(-1)^0 k^m + \binom{m}{1} n^{m-1}(-1)^1 k^{m+1} + \cdots + \binom{m}{m} n^0(-1)^m k^{2m}\right] \\
= \quad & n^{2m+1}.
\end{aligned}$$

Let rewrite expression (4.4) again and move sigma notation under the brackets,

$$
\begin{aligned}
(4.5)\quad n^{2m+1} &= A_{m,0}\sum_{k=1}^{n} k^0(n-k)^0 \ + \cdots + \ A_{m,m}\sum_{k=1}^{n} k^m(n-k)^m \\
&= A_{m,0}\left[\sum_{k=1}^{n}\binom{0}{0}n^0(-1)^0k^0\right] + A_{m,1}\left[\sum_{k=1}^{n}\binom{1}{0}n^1(-1)^0k^1 + \sum_{k=1}^{n}\binom{1}{1}n^0(-1)^1k^2\right] \\
&+ A_{m,2}\left[\sum_{k=1}^{n}\binom{2}{0}n^2(-1)^0k^2 + \sum_{k=1}^{n}\binom{2}{1}n^1(-1)^1k^3 + \sum_{k=1}^{n}\binom{2}{2}n^0(-1)^2k^4\right] \\
&+ A_{m,3}\left[\sum_{k=1}^{n}\binom{3}{0}n^3(-1)^0k^3 + \sum_{k=1}^{n}\binom{3}{1}n^2(-1)^1k^4 + \sum_{k=1}^{n}\binom{3}{2}n^1(-1)^2k^5 + \sum_{k=1}^{n}\binom{3}{3}n^0(-1)^3k^6\right] \\
&\vdots \\
&+ A_{m,m}\left[\sum_{k=1}^{n}\binom{m}{0}n^m(-1)^0k^m + \sum_{k=1}^{n}\binom{m}{1}n^{m-1}(-1)^1k^{m+1} + \cdots + \sum_{k=1}^{n}\binom{m}{m}n^0(-1)^mk^{2m}\right]
\end{aligned}
$$

For example, consider the polynomial $\sum_{k=0}^{m}(-1)^{m-k}U_m(n,k)n^k$, where $m=3$. We derive below a set of coefficients $U_3(n,0),\ U_3(n,1),\ U_3(n,2),\ U_3(n,3)$

$$
\begin{aligned}
(4.6)\quad & \sum_{k=0}^{3}(-1)^{3-k}U_3(n,k)n^k \\
\equiv\ & A_{3,0}\left[\sum_{k=1}^{n}\binom{0}{0}n^0(-1)^0k^0\right] + A_{3,1}\left[\sum_{k=1}^{n}\binom{1}{0}n^1(-1)^0k^1 + \sum_{k=1}^{n}\binom{1}{1}n^0(-1)^1k^2\right] \\
+\ & A_{3,2}\left[\sum_{k=1}^{n}\binom{2}{0}n^2(-1)^0k^2 + \sum_{k=1}^{n}\binom{2}{1}n^1(-1)^1k^3 + \sum_{k=1}^{n}\binom{2}{2}n^0(-1)^2k^4\right] \\
+\ & A_{3,3}\left[\sum_{k=1}^{n}\binom{3}{0}n^3(-1)^0k^3 + \sum_{k=1}^{n}\binom{3}{1}n^2(-1)^1k^4 + \sum_{k=1}^{n}\binom{3}{2}n^1(-1)^2k^5 + \sum_{k=1}^{n}\binom{3}{3}n^0(-1)^3k^6\right] \\
\equiv\ & n^7.
\end{aligned}
$$

The coefficients $U_3(n,0),\ U_3(n,1),\ U_3(n,2),\ U_3(n,3)$ in expression (4.6) are following

$$
\begin{aligned}
(4.7)\qquad U_3(n,0) &= A_{3,0}\sum_{k=1}^{n}\binom{0}{0}(-1)^0k^0 + A_{3,1}\sum_{k=1}^{n}\binom{1}{1}(-1)^1k^2 \\
&+ A_{3,2}\sum_{k=1}^{n}\binom{2}{2}(-1)^2k^4 + A_{3,3}\sum_{k=1}^{n}\binom{3}{3}(-1)^3k^6, \\
(4.8)\qquad U_3(n,1) &= A_{3,1}\sum_{k=1}^{n}\binom{1}{0}(-1)^0k^1 + A_{3,2}\sum_{k=1}^{n}\binom{2}{1}(-1)^1k^3 \\
&+ A_{3,3}\sum_{k=1}^{n}\binom{3}{2}(-1)^2k^5,
\end{aligned}
$$

$$U_3(n,2) = A_{3,2}\sum_{k=1}^{n}\binom{2}{0}(-1)^0k^2 + A_{3,3}\sum_{k=1}^{n}\binom{3}{1}(-1)^1k^4 \tag{4.9}$$

$$U_3(n,3) = A_{3,3}\sum_{k=1}^{n}\binom{3}{0}(-1)^0k^3. \tag{4.10}$$

Let rewrite above identities in compact form as,

$$\begin{aligned} U_3(n,r=0) &= \sum_{t\geq 0}^{3} A_{m,t}\sum_{\ell=1}^{n}\binom{t}{t}(-1)^t\ell^{2t} \\ &= \sum_{t\geq 0}^{3}\sum_{\ell=1}^{n} A_{m,t}\binom{t}{t}(-1)^t\ell^{2t} \end{aligned} \tag{4.11}$$

$$\begin{aligned} U_3(n,r=1) &= \sum_{t\geq 1}^{3} A_{m,t}\sum_{\ell=1}^{n}\binom{t}{t-1}(-1)^{t-1}\ell^{2t-1} \\ &= \sum_{t\geq 1}^{3}\sum_{\ell=1}^{n} A_{m,t}\binom{t}{t-1}(-1)^{t-1}\ell^{2t-1} \end{aligned} \tag{4.12}$$

$$\begin{aligned} U_3(n,r=2) &= \sum_{t\geq 2}^{3} A_{m,t}\sum_{\ell=1}^{n}\binom{t}{t-2}(-1)^{t-2}\ell^{2t-2} \\ &= \sum_{t\geq 2}^{3}\sum_{\ell=1}^{n} A_{m,t}\binom{t}{t-2}(-1)^{t-2}\ell^{2t-2} \end{aligned} \tag{4.13}$$

$$\begin{aligned} U_3(n,r=3) &= \sum_{t\geq 3}^{3} A_{m,t}\sum_{\ell=1}^{n}\binom{t}{t-3}(-1)^{t-3}\ell^{2t-3} \\ &= A_{m,3}\sum_{\ell=1}^{n}\binom{3}{t-3}(-1)^{0}\ell^{3} \end{aligned} \tag{4.14}$$

Note that above compact forms are depend on $r = 0, 1, 2, 3,$ as it can be seen by corresponding binomial coefficients, thus we have replaced k by ℓ, to avoid confusion in notations. Therefore, for every $0 \leq r \leq m$ the $U_m(n,r)$ can be defined next way

$$\begin{aligned} U_m(n,r) &\overset{\text{def}}{=} \sum_{t\geq r}^{m} A_{m,t}\sum_{\ell=1}^{n}\binom{t}{t-r}(-1)^{t-r}\ell^{2t-r} \\ &= \sum_{t\geq r}^{m}\sum_{\ell=1}^{n} A_{m,t}\binom{t}{t-r}(-1)^{t-r}\ell^{2t-r} \end{aligned} \tag{4.15}$$

Similarly, the coefficients $U_m(n-1,r)$ derived just by changing the iteration limits within the sum $\sum_{\ell=1}^{n} A_{m,t}\binom{t}{t-r}(-1)^{t-r}\ell^{2t-r}$ in (4.15) from $\ell \in \{1, n\}$ to $\ell \in \{0, n-1\}$

$$\begin{aligned} U_m(n-1,r) &\overset{\text{def}}{=} \sum_{t\geq r}^{m} A_{m,t}\sum_{\ell=0}^{n-1}\binom{t}{t-r}(-1)^{t-r}\ell^{2t-r} \\ &= \sum_{t\geq r}^{m}\sum_{\ell=0}^{n-1} A_{m,t}\binom{t}{t-r}(-1)^{t-r}\ell^{2t-r} \end{aligned} \tag{4.16}$$

In above equation (4.16) it must be defined $\ell^0 = 1$ for all ℓ, see [31]. Mathematica implementations [5] [6] of expessions (4.15) and (4.16) are available in [41], [42]. Therefore, for every $n \geq 1$, and $m \geq 0$, we have identity

$$n^{2m+1} = \sum_{0 \leq r \leq m} (-1)^{m-r} U_m(n,r) \cdot n^r \tag{4.17}$$

$$= \sum_{0 \leq r \leq m} (-1)^{m-r} U_m(n-1,r) \cdot n^r \tag{4.18}$$

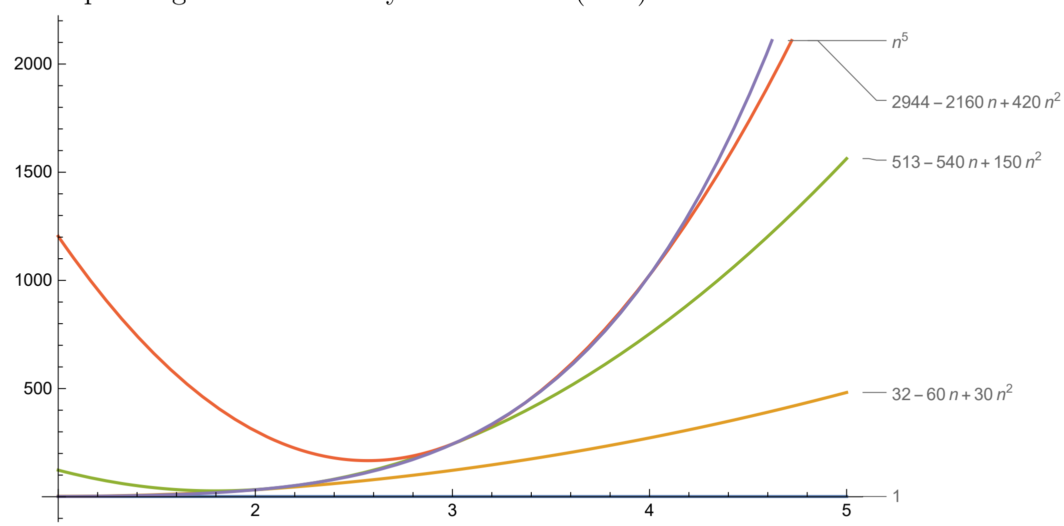
Expressions (4.17), (4.18) are analogs to Faulhaber's odd power identity, see property (2.3) and [21], p. 9. Expressions (4.17), (4.18) could be verified via Mathematica codes [7] [8]. Here we'd like to show a few examples of polynomials generated by $\sum_{0 \leq k \leq T-1} D_2(n,k)$, see definition (3.17) for $D_2(n,k)$

(4.19)

T	$U_2(T-1,0)n^0 + U_2(T-1,1)n^1 + U_2(T-1,2)n^2$
1	1
2	$32 - 60n + 30n^2$
3	$513 - 540n + 150n^2$
4	$2944 - 2160n + 420n^2$
5	$10625 - 6000n + 900n^2$
6	$29376 - 13500n + 1650n^2$
7	$68257 - 26460n + 2730n^2$
8	$140288 - 47040n + 4200n^2$
9	$263169 - 77760n + 6120n^2$
10	$460000 - 121500n + 8550n^2$

Figure 13. Table of polynomials generated by $\sum_{0 \leq k \leq T-1} D_2(n,k)$.

Below we show a plots of few first polynomials $\sum_{1 \leq k \leq T-1} D_2(n,k)$ for $T = 1,2,3,4$ and compare it with corresponding monomial n^5 by the theorem (3.30)



[5] Um(n,k)_coefficients_row_generating.txt. - Mathematica code, implementation of (4.15), prints ten initial rows of coefficients $U_m(n,k)$

[6] Um(n-1,k)_coefficients_row_generating.txt. - Mathematica code, implementation of (4.16), prints ten initial rows of coefficients $U_m(n-1,k)$

[7] Um(n,k)_odd_power_identity.txt - Mathematica code, implementation of (4.17), [37].

[8] Um(n,k)_odd_power_identity2.txt - Mathematica code, implementation of (4.18), [38].

Figure 14. Local approximations of monomial n^5 by corresponding polynomials $\sum_{1\leq k\leq T-1} D_2(n,k)$ for $T=1,2,3,4$, by theorem (3.30).

The polynomials from figure (13) can be generated using Mathematica code[9]. Additionally, the generalized binomial series could be reached by means of identity

$$\begin{aligned}\sum_{M\leq k\leq N} D_1(n,k) &= \begin{cases} U_1(T,0)n^0+U_1(T,1)n^1, & \text{if } M=1,\ N=T \\ U_1(T-1,0)n^0+U_1(T-1,1)n^1, & \text{if } M=0,\ N=T-1 \end{cases} \\ &= n^3,\ \text{as } T\to n.\end{aligned}$$

For instance, for every natural n

$$n^3 = U_1(T,0)\cdot n^0 + U_1(T,1)\cdot n^1$$

To reach higher power m, let just multiply this identity by n^{m-3},

$$n^m = U_1(T,0)\cdot n^{m-2} + U_1(T,1)\cdot n^{m-3}$$

Let rewrite this expression regarding to itself as recursion,

$$\begin{aligned} n^m &= U_1(T,0)\cdot[U_1(T,0)n^{m-4}+U_1(T,1)n^{m-5}] + U_1(T,1)\cdot[U_1(T,0)n^{m-5}+U_1(T,1)n^{m-6}] \\ &= U_1(T,0)^2\cdot n^{m-4} - 2\cdot U_1(T,0)\cdot U_1(T,1)n^{m-5} + U_1(T,1)^2\cdot n^{m-6}\end{aligned}$$

Repeating above process similarly $j\geq 1$ times, we have

$$\begin{aligned} n^m &= \sum_{k\geq 0}(-1)^k\binom{j}{k}U_1(T,0)^{j-k}\cdot U_1(T,1)^k\cdot n^{m-2j-k} \\ &= \sum_{k\geq 0}(-1)^k\binom{j}{k}U_1(T-1,0)^{j-k}\cdot U_1(T-1,1)^k\cdot n^{m-2j-k}\end{aligned} \tag{4.20}$$

We believe we can perform similarly in terms of multinomial coefficients,

$$\begin{aligned} n^m &= \sum_{k_1+k_2+\ldots+k_t=j}\binom{j}{k_1,k_2,\ldots,k_t}\prod_{\ell=1}^{t}(-1)^\ell U_t(T,\ell)^{k_\ell}\cdot n^{m-(t+1)j-k} \\ &= \sum_{k_1+k_2+\ldots+k_t=j}\binom{j}{k_1,k_2,\ldots,k_t}\prod_{\ell=1}^{t}(-1)^\ell U_t(T-1,\ell)^{k_\ell}\cdot n^{m-(t+1)j-k}.\end{aligned} \tag{4.21}$$

Note that we must set $T=n$ in above identities. Another way to represent odd-powered monomial n^{2m+1}, $m=0,1,2,...$ is to define a polynomial $\mathcal{U}_m(n,k)$. As $\sum_{0\leq k\leq m}(-1)^{m-k}U_m(n,k)\cdot n^k = \sum_{0\leq k\leq m}(-1)^{m-k}U_m(n-1,k)\cdot n^k$, it follows that

$$\begin{aligned} 2n^{2m+1} &= \sum_{0\leq k\leq m}(-1)^{m-k}U_m(n,k)\cdot n^k + \sum_{0\leq k\leq m}(-1)^{m-k}U_m(n-1,k)\cdot n^k \\ &= \sum_{0\leq k\leq m}(-1)^{m-k}(U_m(n,k)+U_m(n-1,k))\cdot n^k,\end{aligned}$$

Let be definition

$$\mathcal{U}_m(n,k) \stackrel{\text{def}}{=} \frac{1}{2}\Big[U_m(n,k)+U_m(n-1,k)\Big], \tag{4.22}$$

Therefore,

$$n^{2m+1} = \sum_{0\leq k\leq m}\mathcal{U}_m(n,k)\cdot n^k. \tag{4.23}$$

[9] Um(n,k)_coefficients.txt. - Mathematica code, generates the polynomials from figure (12), [40]. If necessary, one could re-define $m\geq 0$ within the code.

Expression (4.23) generates the following polynomials given $m = 2$,

(4.24)

T	$\mathcal{U}_2(T,0)n^0 + \mathcal{U}_2(T,1)n^1 + \mathcal{U}_2(T,2)n^2$
1	$16 - 30n + 15n^2$
2	$272 - 300n + 90n^2$
3	$1728 - 1350n + 285n^2$
4	$6784 - 4080n + 660n^2$
5	$20000 - 9750n + 1275n^2$
6	$48816 - 19980n + 2190n^2$
7	$104272 - 36750n + 3465n^2$
8	$201728 - 62400n + 5160n^2$
9	$361584 - 99630n + 7335n^2$
10	$610000 - 151500n + 10050n^2$

Figure 15. Table of polynomials generated by $\sum_{0 \le k \le m} \mathcal{U}_m(T,k) \cdot n^k$, where $T = 1, 2, 3, .., 10$.

Polynomials from Figure (15) could be generated using Mathematica code[10]. Let graphically show an approximation of monomial n^5 by polynomials $\sum_{0 \le k \le m} \mathcal{U}_2(T,k) \cdot n^k$ for $T = 1, 2, 3, 4$.

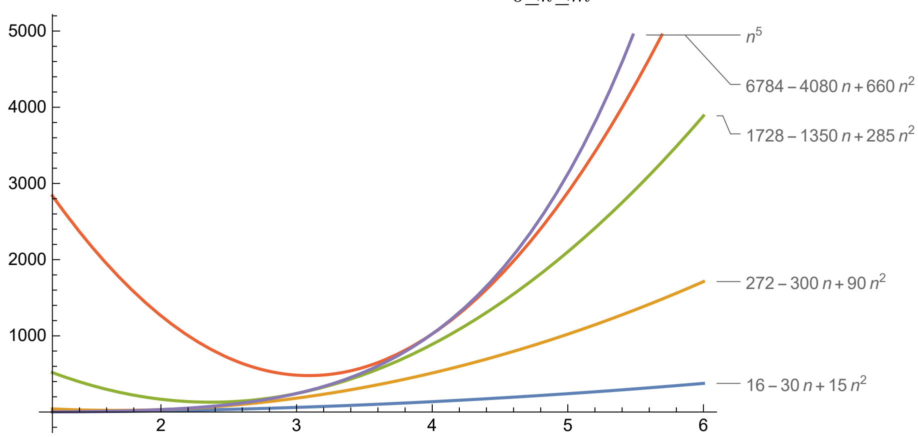


Figure 16. Local approximations of monomial n^5 by corresponding polynomials $\sum_{0 \le k \le m} \mathcal{U}_2(T,k) \cdot n^k$ for $T = 1, 2, 3, 4$.

Definition (4.22) of coefficients $\mathcal{U}_m(n,k)$ and identity (4.23) could be verified via Mathematica codes [11] [12], respectively. Similarly to (4.20), (4.21) we can perform a binomial and multinomial representation of monomial n^m, where n, m are non-negative integers and $T = n$,

$$n^m = \sum_{k \ge 0} (-1)^k \binom{j}{k} \mathcal{U}_1(T,0)^{j-k} \cdot \mathcal{U}_1(T,1)^k \cdot n^{m-2j-k} \tag{4.25}$$

$$= \sum_{k \ge 0} (-1)^k \binom{j}{k} \mathcal{U}_1(T-1,0)^{j-k} \cdot \mathcal{U}_1(T-1,1)^k \cdot n^{m-2j-k}$$

[10] Combined_U_m(n,k)_coefficients_polynomials_gf.txt. - Mathematica code, generates the polynomials from Figure (15), [43].

[11] Combined_U_m(n,k)_coefficients_gf.txt. - Mathematica code, verifies the values definition (4.22) produces, [44].

[12] Combined_U_m(n,k)_coefficients_odd_power_identity.txt. - Mathematica code, verifies identity (4.23), [45].

And

$$n^m = \sum_{k_1+k_2+\ldots+k_t=j} \binom{j}{k_1, k_2, \ldots, k_t} \prod_{\ell=1}^{t} (-1)^{\ell} \mathcal{U}_t(T,\ell)^{k_\ell} \cdot n^{m-(t+1)j-k} \tag{4.26}$$

$$= \sum_{k_1+k_2+\ldots+k_t=j} \binom{j}{k_1, k_2, \ldots, k_t} \prod_{\ell=1}^{t} (-1)^{\ell} \mathcal{U}_t(T-1,\ell)^{k_\ell} \cdot n^{m-(t+1)j-k}.$$

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6. Conclusion

In this paper particular pattern, that is binomial distributed triangle A287326 in OEIS, which shows perfect cube n as sum of row terms over $0 \leq k \leq n-1$ or $1 \leq k \leq n$ is generalized, in this manuscript are found and discussed the polynomials $D_m(n,k)$ and $U_m(n,k)$, such that, when being summed up over k in some range with respect to m and n returns the monomial n^{2m+1}. As first step, we discussed analogs of A287326 for powers $l = 5, 7$, sequences A300656, A300785, respectively, then we derived coefficients $A_{m,j}$, such that for every $n \geq 0$ and $m \geq 0$ holds

$$n^{2m+1} = \sum_{1 \leq k \leq n} D_m(n,k),$$

where

$$D_m(n,k) := A_{m,m} k^m (n-k)^m + A_{m,m-1} k^{m-1} (n-k)^{m-1} + \cdots + A_{m,0} k^0 (n-k)^0,$$

and $A_{m,j}$ is defined by (3.28). Therefore, question question (2.23) is answered positively. Section (3) is totally dedicated to complete and extended derivation of identity $\sum_{1 \leq k \leq n} D_m(n,k) = n^{2m+1}$. Properties of triangle (2.14) and polynomial $D_m(n,k)$ are shown in properties 2.15 and subsection 3.1, respectively. Relation between Faulhaber's sum $\sum n^m$ and finite differences of power are shown in 2.3. In section 4 we have generalized the main result of Corollary (2.2) for all odd powers of the form $2m+1$, $m = 0, 1, 2, \ldots$ and proven an identities

$$n^{2m+1} = \sum_{0 \leq r \leq m} (-1)^{m-r} U_m(n,r) \cdot n^r$$

$$= \sum_{0 \leq r \leq m} (-1)^{m-r} U_m(n-1,r) \cdot n^r.$$

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7. Application 1. Extended tables of polynomials, consisting $U_m(n,k)$ coefficients

In this application we attach an extended table, consisting of polynomials, generated by $\sum_{1\leq k\leq T} D_m(n,k)$ and $\sum_{0\leq k\leq T-1} D_m(n,k)$ for various t and m. We begin from cases $m = 2, 3, 4$ given generating function $\sum_{1\leq k\leq T} D_m(n,k)$ and continue similarly with examples for $\sum_{0\leq k\leq T-1} D_m(n,k)$. The following tables could be generated using Mathematica code Um(n,k)_coefficients2.txt. Here we begin to show our tables for $m = 1, 2, 3, 4$ and $T = 1, 2, ..., 40$

t	Polynomial(n)
1	$31 - 60n + 30n^2$
2	$512 - 540n + 150n^2$
3	$2943 - 2160n + 420n^2$
4	$10624 - 6000n + 900n^2$
5	$29375 - 13500n + 1650n^2$
6	$68256 - 26460n + 2730n^2$
7	$140287 - 47040n + 4200n^2$
8	$263168 - 77760n + 6120n^2$
9	$459999 - 121500n + 8550n^2$
10	$760000 - 181500n + 11550n^2$
11	$1199231 - 261360n + 15180n^2$
12	$1821312 - 365040n + 19500n^2$
13	$2678143 - 496860n + 24570n^2$
14	$3830624 - 661500n + 30450n^2$
15	$5349375 - 864000n + 37200n^2$
16	$7315456 - 1109760n + 44880n^2$
17	$9821087 - 1404540n + 53550n^2$
18	$12970368 - 1754460n + 63270n^2$
19	$16879999 - 2166000n + 74100n^2$
20	$21680000 - 2646000n + 86100n^2$
21	$27514431 - 3201660n + 99330n^2$
22	$34542112 - 3840540n + 113850n^2$
23	$42937343 - 4570560n + 129720n^2$
24	$52890624 - 5400000n + 147000n^2$
25	$64609375 - 6337500n + 165750n^2$
26	$78318656 - 7392060n + 186030n^2$
27	$94261887 - 8573040n + 207900n^2$
28	$112701568 - 9890160n + 231420n^2$
29	$133919999 - 11353500n + 256650n^2$
30	$158220000 - 12973500n + 283650n^2$

Figure 17. Table for $m = 2$, generating function: $\sum_{1\leq k\leq T} D_m(n,k)$ over $T = 1, 2, ..., 30$.

t	Polynomial(n)
1	$-125 + 406n - 420n^2 + 140n^3$
2	$-9028 + 13818n - 7140n^2 + 1260n^3$

3	$-110961+115836n-41160n^2+5040n^3$
4	$-684176+545860n-148680n^2+14000n^3$
5	$-2871325+1858290n-411180n^2+31500n^3$
6	$-9402660+5124126n-955500n^2+61740n^3$
7	$-25872833+12182968n-1963920n^2+109760n^3$
8	$-62572096+25945416n-3684240n^2+181440n^3$
9	$-136972701+50745870n-6439860n^2+283500n^3$
10	$-276971300+92745730n-10639860n^2+423500n^3$
11	$-524988145+160386996n-16789080n^2+609840n^3$
12	$-943023888+264896268n-25498200n^2+851760n^3$
13	$-1618774781+420839146n-37493820n^2+1159340n^3$
14	$-2672907076+646725030n-53628540n^2+1543500n^3$
15	$-4267591425+965662320n-74891040n^2+2016000n^3$
16	$-6616398080+1406064016n-102416160n^2+2589440n^3$
17	$-9995653693+2002403718n-137494980n^2+3277260n^3$
18	$-14757360516+2796022026n-181584900n^2+4093740n^3$
19	$-21343778801+3835983340n-236319720n^2+5054000n^3$
20	$-30303773200+5179983060n-303519720n^2+6174000n^3$
21	$-42311023965+6895305186n-385201740n^2+7470540n^3$
22	$-58184203748+9059830318n-483589260n^2+8961260n^3$
23	$-78909220801+11763094056n-601122480n^2+10664640n^3$
24	$-105663629376+15107395800n-740468400n^2+12600000n^3$
25	$-139843308125+19208957950n-904530900n^2+14787500n^3$
26	$-183091507300+24199135506n-1096460820n^2+17248140n^3$
27	$-237330365553+30225676068n-1319666040n^2+20003760n^3$
28	$-304794997136+37454030236n-1577821560n^2+23077040n^3$
29	$-388070250301+46068712410n-1874879580n^2+26491500n^3$
30	$-490130237700+56274711990n-2215079580n^2+30271500n^3$
31	$-614380739585+68298954976n-2602958400n^2+34442240n^3$
32	$-764704580608+82391815968n-3043360320n^2+39029760n^3$
33	$-945510081021+98828680566n-3541447140n^2+44060940n^3$
34	$-1161782683076+117911558170n-4102708260n^2+49563500n^3$
35	$-1419139853425+139970745180n-4732970760n^2+55566000n^3$
36	$-1723889362320+165366538596n-5438409480n^2+62097840n^3$
37	$-2083091040413+194491000018n-6225557100n^2+69189260n^3$
38	$-2504622113956+227769770046n-7101314220n^2+76871340n^3$
39	$-2997246219201+265663933080n-8072959440n^2+85176000n^3$
40	$-3570686196800+308671932520n-9148159440n^2+94136000n^3$

Figure 18. For $m=3$, generating function: $\sum_{1\le k\le T} D_m(n,k)$ over $T=1,2,...,40$.

t	Polynomial(n)
1	$751-2640n+3780n^2-2520n^3+630n^4$
2	$162512-325440n+245700n^2-83160n^3+10710n^4$
3	$4297023-5837040n+3001320n^2-695520n^3+61740n^4$
4	$45586624-47125200n+18484200n^2-3276000n^3+223020n^4$
5	$291683375-244000800n+77546700n^2-11151000n^3+616770n^4$
6	$1349845776-949440240n+253906380n^2-30746520n^3+1433250n^4$
7	$4981676287-3024769440n+698619600n^2-73100160n^3+2945880n^4$
8	$15551330048-8309593440n+1689523920n^2-155675520n^3+5526360n^4$

9	$42670773999 - 20362676400n + 3698370900n^2 - 304479000n^3 + 9659790n^4$
10	$105670786000 - 45562677600n + 7478370900n^2 - 556479000n^3 + 15959790n^4$
11	$240716895551 - 94670349840n + 14174871480n^2 - 962327520n^3 + 25183620n^4$
12	$511605381312 - 184966507440n + 25461891000n^2 - 1589384160n^3 + 38247300n^4$
13	$1025515755823 - 343092771840n + 43707229020n^2 - 2525042520n^3 + 56240730n^4$
14	$1955262884624 - 608734803600n + 72168875100n^2 - 3880359000n^3 + 80442810n^4$
15	$3569884005375 - 1039300430400n + 115225437600n^2 - 5793984000n^3 + 112336560n^4$
16	$6275713432576 - 1715757781440n + 178643314080n^2 - 8436395520n^3 + 153624240n^4$
17	$10670440655087 - 2749811239440n + 269883324900n^2 - 12014435160n^3 + 206242470n^4$
18	$17613015856848 - 4292605722240n + 398449531620n^2 - 16776146520n^3 + 272377350n^4$
19	$28312660615999 - 6545162506800n + 576282961800n^2 - 23015916000n^3 + 354479580n^4$
20	$44440660664000 - 9770762509200n + 818202961800n^2 - 31079916000n^3 + 455279580n^4$
21	$68269062114351 - 14309505635040n + 1142398899180n^2 - 41371850520n^3 + 577802610n^4$
22	$102840862500112 - 20595287515440n + 1570974936300n^2 - 54359003160n^3 + 725383890n^4$
23	$152176783290623 - 29175447644640n + 2130550596720n^2 - 70578587520n^3 + 901683720n^4$
24	$221524231290624 - 40733355636000n + 2852919846000n^2 - 90644400000n^3 + 1110702600n^4$
25	$317654602459375 - 56114215014000n + 3775771408500n^2 - 115253775000n^3 + 1356796350n^4$
26	$449215653223376 - 76354376660640n + 4943473041780n^2 - 145194842520n^3 + 1644691230n^4$
27	$627146261293887 - 102714466735440n + 6407922490200n^2 - 181354088160n^3 + 1979499060n^4$
28	$865161520339648 - 136716646589040n + 8229467839320n^2 - 224724215520n^3 + 2366732340n^4$
29	$1180316760605999 - 180186334891200n + 10477899992700n^2 - 276412311000n^3 + 2812319370n^4$
30	$1593659760714000 - 235298734894800n + 13233519992700n^2 - 337648311000n^3 + 3322619370n^4$
31	$2130981114417151 - 304630522458240n + 16588283906880n^2 - 409793771520n^3 + 3904437600n^4$
32	$2823673440038912 - 391217063149440n + 20647028001600n^2 - 494350940160n^3 + 4565040480n^4$
33	$3709710869661423 - 498615539455440n + 25528776924420n^2 - 592972130520n^3 + 5312170710n^4$
34	$4834761029884624 - 630974381822400n + 31368137616900n^2 - 707469399000n^3 + 6154062390n^4$
35	$6253442526125375 - 793109409951600n + 38316781679400n^2 - 839824524000n^3 + 7099456140n^4$
36	$8030741767978176 - 990587103477840n + 46545018909480n^2 - 992199287520n^3 + 8157614220n^4$
37	$10243603824112687 - 1229815433857440n + 56243464735500n^2 - 1166946059160n^3 + 9338335650n^4$
38	$12982712871538448 - 1518142701993840n + 67624804267020n^2 - 1366618682520n^3 + 10651971330n^4$
39	$16354478705823999 - 1863964838829600n + 80925655683600n^2 - 1593983664000n^3 + 12109439160n^4$
40	$20483246706016000 - 2276841638834400n + 96408535683600n^2 - 1852031664000n^3 + 13722239160n^4$

Figure 18. For $m = 4$, generating function: $\sum_{1 \le k \le T} D_m(n, k)$ over $T = 1, 2, ..., 40$.

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